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n-Dimensional Representations of the Unified Field Tensor $*g^{\lambda\nu}$

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For the first time Hlavatý represented the tensor $*g^{\lambda\nu}$, defined by (15a), in terms of the unified field tensor $g_{\lambda\mu}$ in the space-time X_4. Recently, the representations of $*g^{\lambda\nu}$ in terms of $g_{\lambda\mu}$ in two- and three-dimensional generalized Riemannian space were obtained by Chung. The purpose of the present paper is to obtain the generalized representations of $*g^{\lambda\nu}$ in terms of $g_{\lambda\mu}$ in a generalized n-dimensional Riemannian space X_n.

1. INTRODUCTION

The tensor $*g^{\lambda\nu}$ defined by (15) is very useful, particularly for the study of Einstein's unified field theory to physical applications. For the first time Hlavatý represented the tensor $*g^{\lambda\nu}$, defined by (15a), in terms of the unified field tensor $g_{\lambda\mu}$ in the space–time X_4 (Hlavatý, 1957). The representations of $*g^{\lambda\nu}$ in terms of $g_{\lambda\mu}$ in two- and three-dimensional generalized Riemannian space were obtained recently by Chung (1979). The purpose of the present paper is to obtain the generalized representations of $*g^{\lambda\nu}$ in terms of $g_{\lambda\mu}$ in a generalized n-dimensional Riemannian space X_n. The obtained results and discussions in the present paper will be useful for the n-dimensional considerations of $*g^{\lambda\nu}$-unified field theory.

2. PRELIMINARY RESULTS

This section is a collection of notations and basic results which will be needed in our subsequent considerations. The detailed proofs are given in Chung (1975).

In the usual Einstein's n-dimensional unified field theory, the generalized n-dimensional Riemannian space X_n refers to a real coordinate transformation $x^\lambda \to \bar{x}^\lambda$, for which

$$\mathrm{Det}\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0 \tag{1}$$

and is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2}$$

Here the matrices $(g_{\lambda\mu})$ and $(h_{\lambda\mu})$ are assumed to be of rank n. We may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu \tag{3}$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner.

In our subsequent considerations, the following densities, scalars, and tensors are frequently used:

$$\mathfrak{G} \overset{\text{def}}{=} \mathrm{Det}(g_{\lambda\mu}), \qquad \mathfrak{H} \overset{\text{def}}{=} \mathrm{Det}(h_{\lambda\mu}), \qquad \mathfrak{T} \overset{\text{def}}{=} \mathrm{Det}(k_{\lambda\mu}) \tag{4a}$$

$$g \overset{\text{def}}{=} \mathfrak{G}/\mathfrak{H}, \qquad k \overset{\text{def}}{=} \mathfrak{T}/\mathfrak{H} \tag{4b}$$

$$^{(0)}k_\lambda{}^\nu \overset{\text{def}}{=} \delta_\lambda^\nu, \qquad ^{(p)}k_\lambda{}^\nu \overset{\text{def}}{=} {}^{(p-1)}k_\lambda{}^\mu k_\mu{}^\nu \qquad (p=1,2,\ldots). \tag{4c}$$

The n-dimensional contravariant and covariant indicators, $E^{\alpha_1 \cdots \alpha_n}$ and $e_{\alpha_1 \cdots \alpha_n}$, satisfy the following identities:

$$E_{\alpha_1 \cdots \alpha_n} = \mathfrak{H} e_{\alpha_1 \cdots \alpha_n}, \qquad \mathfrak{H} e^{\alpha_1 \cdots \alpha_n} = E^{\alpha_1 \cdots \alpha_n} \tag{5}$$

$$\delta^{\lambda\alpha_1 \cdots \alpha_p}_{[\lambda\beta_1 \cdots \beta_p]} = \frac{n-p}{p+1} \delta^{\alpha_1 \cdots \alpha_p}_{[\beta_1 \cdots \beta_p]} \qquad (p=0,1,2,\ldots,n-1) \tag{6}$$

$$E^{\alpha_1 \cdots \alpha_p \alpha_{p+1} \cdots \alpha_n} e_{\beta_1 \cdots \beta_p \alpha_{p+1} \cdots \alpha_n} = p!(n-p)!\delta^{\alpha_1 \cdots \alpha_p}_{[\beta_1 \cdots \beta_p]} \qquad (p=0,1,2,\ldots,n) \tag{7}$$

Using (5), (6), and (7), we may easily verify that the scalars M_p and K_p, defined by

$$M_p \stackrel{\text{def}}{=} E^{\alpha_1 \cdots \alpha_p \alpha_{p+1} \cdots \alpha_n} E^{\beta_1 \cdots \beta_p \beta_{p+1} \cdots \beta_n} k_{\alpha_1 \beta_1} \cdots k_{\alpha_p \beta_p} h_{\alpha_{p+1} \beta_{p+1}} \cdots h_{\alpha_n \beta_n} \tag{8a}$$

$$K_p \stackrel{\text{def}}{=} k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \cdots k_{\alpha_p]}{}^{\alpha_p} \tag{8b}$$

satisfy the following relations for $p=0,1,2,\ldots,n$:

$$M_p = p!(n-p)!\mathfrak{H} K_p \tag{9}$$

As a direct consequence of (9) and the basic polynomial of X_n, we have

$$M_0 = n!\mathfrak{H}, \qquad M_n = n!\mathfrak{T} \tag{10a}$$

$$K_0 = 1, \qquad K_n = k \qquad \text{if } n \text{ is even} \tag{10b}$$

$$M_p = K_p = 0 \qquad \text{if } p \text{ is odd} \tag{10c}$$

$$\mathfrak{G} = \mathfrak{H} \sum_{p=0}^{n} K_p = \sum_{p=0}^{n} \frac{1}{p!(n-p)!} M_p \tag{11}$$

3. THE TENSORS $\underset{(s)}{Z}{}_\lambda{}^\nu$ AND $\underset{(s)}{X}{}_\lambda{}^\nu$

In this section we shall derive useful representations of two tensors $\underset{(s)}{Z}{}_\lambda{}^\nu$ and $\underset{(s)}{X}{}_\lambda{}^\nu$, which will be needed in our considerations of the next section.

Put

$$\underset{(0)}{Z}{}_\lambda{}^\nu \stackrel{\text{def}}{=} k_\lambda{}^\nu, \qquad \underset{(s)}{Z}{}_\lambda{}^\nu \stackrel{\text{def}}{=} k_{[\alpha_1}{}^\nu k_{\alpha_2}{}^{\alpha_1} \cdots k_{\alpha_s}{}^{\alpha_{s-1}} k_{\lambda]}{}^{\alpha_s}, \tag{12a}$$

$$\underset{(s)}{X}{}_\lambda{}^\nu \stackrel{\text{def}}{=} \delta^{\alpha_1 \cdots \alpha_s \nu}_{[\beta_1 \cdots \beta_s \lambda]} k_{\alpha_1}{}^{\beta_1} \cdots k_{\alpha_s}{}^{\beta_s}, \qquad (s=0,1,2,\ldots,n-1) \tag{12b}$$

Direct calculations employing the notations in (8) are shown in Table I.

TABLE I. The Tensors $\underset{(s)}{Z_\lambda}{}^\nu$ and $\underset{(s)}{X_\lambda}{}^\nu$ for $s=0,1,2,3,4$

s	$\underset{(s)}{Z_\lambda}{}^\nu$	$\underset{(s)}{X_\lambda}{}^\nu$
0	$k_\lambda{}^\nu$	δ_λ^ν
1	$(1/2)^{(2)}k_\lambda{}^\nu$	$-(1/2)k_\lambda{}^\nu$
2	$(1/3)(K_2k_\lambda{}^\nu+{}^{(3)}k_\lambda{}^\nu)$	$(1/3)(K_2\delta_\lambda^\nu+{}^{(2)}k_\lambda{}^\nu)$
3	$(1/4)(K_2^{(2)}k_\lambda{}^\nu+{}^{(4)}k_\lambda{}^\nu)$	$-(1/4)(K_2k_\lambda{}^\nu+{}^{(3)}k_\lambda{}^\nu)$
4	$(1/5)(K_4k_\lambda{}^\nu+K_2{}^{(3)}k_\lambda{}^\nu+{}^{(5)}k_\lambda{}^\nu)$	$(1/5)(K_4\delta_\lambda^\nu+K_2{}^{(2)}k_\lambda{}^\nu+{}^{(4)}k_\lambda{}^\nu)$

Theorem 3.1. The tensor $\underset{(s)}{Z_\lambda}{}^\nu$ may be given by

$$\underset{(s)}{Z_\lambda}{}^\nu=\frac{1}{s+1}\sum_{r=0}^{s}(-1)^{s-r}K_{s-r}{}^{(r+1)}k_\lambda{}^\nu \tag{13a}$$

which is equivalent to

$$\underset{(s)}{Z_\lambda}{}^\nu=\begin{cases}\dfrac{1}{s+1}\left(K_0{}^{(s+1)}k_\lambda{}^\nu+K_2{}^{(s-1)}k_\lambda{}^\nu+\cdots K_{s-2}{}^{(3)}k_\lambda{}^\nu+K_sk_\lambda{}^\nu\right) & (s\text{ even})\\[2ex] \dfrac{1}{s+1}\left(K_0{}^{(s+1)}k_\lambda{}^\nu+K_2{}^{(s-1)}k_\lambda{}^\nu+\cdots K_{s-3}{}^{(4)}k_\lambda{}^\nu+K_{s-1}{}^{(2)}k_\lambda{}^\nu\right) & (s\text{ odd})\end{cases} \tag{13b}$$

Proof. This assertion will be proved by induction on s. By virtue of (10) and Table I, it can be easily seen that the assertion holds for the cases $s=0,1,2,3,4$. Now, assume that (13a) is true for the case $s=m-1$ $(<n-1)$, i.e.,

$$\underset{(m-1)}{Z_\lambda}{}^\nu=\frac{1}{m}\sum_{r=0}^{m-1}(-1)^{m-1-r}K_{m-1-r}{}^{(r+1)}k_\lambda{}^\nu \tag{13c}$$

Then, according to the above inductive hypothesis, we have

$$\underset{(m)}{Z_\lambda}{}^\nu=\begin{cases}\dfrac{m!}{(m+1)!}\Big(k_\lambda{}^\nu k_{[\alpha_1}{}^{\alpha_1}\cdots k_{\alpha_m]}{}^{\alpha_m}+k_{\alpha_1}{}^\nu k_{[\alpha_2}{}^{\alpha_1}\cdots k_{\alpha_m}{}^{\alpha_{m-1}}k_{\lambda]}{}^{\alpha_m}\\ \qquad +k_{\alpha_2}{}^\nu k_{[\alpha_3}{}^{\alpha_1}\cdots k_\lambda{}^{\alpha_{m-1}}k_{\alpha_1]}{}^{\alpha_m}\\ \qquad +\cdots+k_{\alpha_m}{}^\nu k_{[\lambda}{}^{\alpha_1}k_{\alpha_1}{}^{\alpha_2}\cdots k_{\alpha_{m-1}]}{}^{\alpha_m}\Big), & \text{if } m \text{ is even}\\[2ex] \dfrac{m!}{(m+1)!}\Big(-k_\lambda{}^\nu k_{[\alpha_1}{}^{\alpha_1}\cdots k_{\alpha_m]}{}^{\alpha_m}+k_{\alpha_1}{}^\nu k_{[\alpha_2}{}^{\alpha_1}\cdots k_{\alpha_m}{}^{\alpha_{m-1}}k_{\lambda]}{}^{\alpha_m}\\ \qquad -k_{\alpha_2}{}^\nu k_{[\alpha_3}{}^{\alpha_1}\cdots k_\lambda{}^{\alpha_{m-1}}k_{\alpha_1]}{}^{\alpha_m}\\ \qquad +\cdots+k_{\alpha_m}{}^\nu k_{[\lambda}{}^{\alpha_1}k_{\alpha_1}{}^{\alpha_2}\cdots k_{\alpha_{m-1}]}{}^{\alpha_m}\Big), & \text{if } m \text{ is odd}\end{cases}$$

$$=\frac{1}{m+1}\left[(-1)^m K_m k_\lambda{}^\nu + m k_{\alpha_1}{}^\nu k_{[\alpha_2}{}^{\alpha_1}\cdots k_{\alpha_m}{}^{\alpha_{m-1}} k_{\lambda]}{}^{\alpha_m}\right]$$

$$=\frac{1}{m+1}\left[(-1)^m K_m k_\lambda{}^\nu + m k_{\alpha_1}{}^\nu \underset{(m-1)}{Z_\lambda{}^{\alpha_1}}\right]$$

$$=\frac{1}{m+1}\left[(-1)^m K_m k_\lambda{}^\nu + \sum_{r=0}^{m-1}(-1)^{m-1-r} K_{m-1-r}{}^{(r+2)}k_\lambda{}^\nu\right]$$

$$=\frac{1}{m+1}\sum_{r=0}^{m}(-1)^{m-r} K_{m-r}{}^{(r+1)}k_\lambda{}^\nu$$

which shows that (13a) holds for the case $s=m$. The equivalence of (13b) to (13a) follows from (10c). ■

Theorem 3.2. The tensor $\underset{(s)}{X_\lambda{}^\nu}$ may be given by

$$\underset{(s)}{X_\lambda{}^\nu}=\frac{1}{s+1}\sum_{r=0}^{s}(-1)^r K_{s-r}{}^{(r)}k_\lambda{}^\nu \tag{14a}$$

which is equivalent to

$$\underset{(s)}{X^{\lambda\nu}}=\begin{cases}\dfrac{1}{s+1}\left(K_0{}^{(s)}k^{\lambda\nu}+K_2{}^{(s-2)}k^{\lambda\nu}+\cdots+K_{s-2}{}^{(2)}k^{\lambda\nu}+K_s h^{\lambda\nu}\right) & (s \text{ even})\\[2mm] \dfrac{1}{s+1}\left(K_0{}^{(s)}k^{\nu\lambda}+K_2{}^{(s-2)}k^{\nu\lambda}+\cdots+K_{s-3}{}^{(3)}k^{\nu\lambda}+K_{s-1}k^{\nu\lambda}\right) & (s \text{ odd})\end{cases} \tag{14b}$$

The tensor $\underset{(s)}{X^{\lambda\nu}}$ is symmetric (skew symmetric) if s is even (odd).

Proof. By using (12a), (12b), (8b), and (13c), the assertion (14a) may be proved as in the following way:

$$\underset{(s)}{X_\lambda{}^\nu}=\frac{s!}{(s+1)!}\left[\delta^{\alpha_1\cdots\alpha_s}_{[\beta_1\cdots\beta_s]}\delta^\nu_\lambda+(-1)^s s\,\delta^{\alpha_1\cdots\alpha_{m-1}\alpha_m}_{[\beta_2\cdots\beta_m\lambda]}\delta^\nu_{\beta_1}\right]k_{\alpha_1}{}^{\beta_1}\cdots k_{\alpha_s}{}^{\beta_s}$$

$$=\frac{1}{s+1}\left[K_s\delta^\nu_\lambda+(-1)^s s k_{[\beta_2}{}^\nu k_{\beta_3}{}^{\beta_2}\cdots k_{\beta_m}{}^{\beta_{m-1}}k_{\lambda]}{}^{\beta_m}\right]$$

$$=\frac{1}{s+1}\left[K_s\delta^\nu_\lambda+(-1)^s s\underset{(s-1)}{Z_\lambda{}^\nu}\right]$$

$$= \frac{1}{s+1}\left[K_s\delta_\lambda{}^\nu + \sum_{r=0}^{s-1}(-1)^{r+1}K_{s-r-1}{}^{(r+1)}k_\lambda{}^\nu\right]$$

$$= \frac{1}{s+1}\sum_{r=0}^{s}(-1)^r K_{s-r}{}^{(r)}k_\lambda{}^\nu$$

(14b) follows from (14a) by virtue of (10c). The last statement follows from the fact that the tensor ${}^{(q)}k^{\lambda\nu}$ is symmetric (skew symmetric) if q is even (odd). ■

Note: As defined in (12a) and (12b), it should be noted that the integers s in the above two theorems take the values $0, 1, \ldots, n-1$.

4. THE TENSOR $^*g^{\lambda\nu}$ IN X_n

In this section we derive useful representations of the unified field tensor $^*g^{\lambda\nu}$ in X_n, uniquely defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu^\nu \tag{15a}$$

which is equivalent to

$$^*g^{\lambda\nu} = \frac{\partial \ln \mathfrak{G}}{\partial g_{\lambda\nu}} \tag{15b}$$

The tensor $^*g^{\lambda\nu}$ may also be decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu} \tag{16}$$

Agreement 4.1. In our further considerations, we assume that the values of p are even integers from 0 to n.

Theorem 4.2. We have

$$\frac{\partial M_p}{\partial h_{\lambda\nu}} = (p+1)!(n-p)!\mathfrak{H}\, \underset{(p)}{X}{}^{\nu\lambda} \tag{17a}$$

$$\frac{\partial M_p}{\partial k_{\lambda\nu}} = pp!(n-p)!\mathfrak{H}\, \underset{(p-1)}{X}{}^{\nu\lambda} \tag{17b}$$

Proof. Using (8a), (5), (7), and (12b), we may derive (17a) as in the following way:

$$\frac{\partial M_p}{\partial h_{\lambda\nu}} = (n-p)E^{\alpha_1\cdots\alpha_{n-1}\lambda}E^{\beta_1\cdots\beta_{n-1}\nu}k_{\alpha_1\beta_1}\cdots k_{\alpha_p\beta_p}h_{\alpha_{p+1}\beta_{p+1}}\cdots h_{\alpha_{n-1}\beta_{n-1}}$$

$$= (n-p)\mathfrak{G}E^{\alpha_1\cdots\alpha_{n-1}\lambda}e_{\beta_1\cdots\beta_p\alpha_{p+1}\cdots\alpha_{n-1}\mu}k_{\alpha_1}{}^{\beta_1}\cdots k_{\alpha_p}{}^{\beta_p}h^{\nu\mu}$$

$$= (n-p)!(p+1)!\mathfrak{G}\delta^{\alpha_1\cdots\alpha_p\lambda}_{[\beta_1\cdots\beta_p\mu]}k_{\alpha_1}{}^{\beta_1}\cdots k_{\alpha_p}{}^{\beta_p}h^{\nu\mu}$$

$$= (n-p)!(p+1)!\mathfrak{G}\underset{(p)}{X}{}^{\nu\lambda}$$

(17b) may be obtained similarly. ■

Remark 4.3. It should be remarked that (17a) and (17b) are general results in X_n, with the following agreements:

$$\underset{(n)}{X}{}^{\lambda\nu} = \underset{(-1)}{X}{}^{\lambda\nu} = 0 \tag{18}$$

Now, we are ready to derive the following representations of the tensor $^*g^{\lambda\nu}$ in terms of $g_{\lambda\mu}$.

Theorem 4.4. The tensors $^*h^{\lambda\nu}$ and $^*k^{\lambda\nu}$ satisfy the following equations:

$$^*h^{\lambda\nu} = \frac{1}{g}\sum_{p=0}^{n-1}(p+1)\underset{(p)}{X}{}^{\nu\lambda} \tag{19a}$$

$$^*k^{\lambda\nu} = \frac{1}{g}\sum_{p=2}^{n} p\,\underset{(p-1)}{X}{}^{\nu\lambda} \tag{20a}$$

These equations are, respectively, equivalent to

$$^*h^{\lambda\nu} = \frac{1}{g}\sum_{p=0}^{n-1}\Big(K_0{}^{(p)}k^{\lambda\nu} + K_2{}^{(p-2)}k^{\lambda\nu} + \cdots$$

$$+ K_{p-2}{}^{(2)}k^{\lambda\nu} + K_p h^{\lambda\nu}\Big) \tag{19b}$$

$$^*k^{\lambda\nu} = \frac{1}{g}\sum_{p=2}^{n}\Big(K_0{}^{(p-1)}k^{\lambda\nu} + K_2{}^{(p-3)}k^{\lambda\nu} + \cdots$$

$$+ K_{p-4}{}^{(3)}k^{\lambda\nu} + K_{p-2}k^{\lambda\nu}\Big) \tag{20b}$$

TABLE II. The Tensors $*h^{\lambda\nu}$ and $*k^{\lambda\nu}$ in X_n for $n=2,3$, and 4

n	$*h^{\lambda\nu}$	$*k^{\lambda\nu}$
2	$(1/g)h^{\lambda\nu}$	$(1/g)k^{\lambda\nu}$
3	$h^{\lambda\nu}+(1/g)^{(2)}k^{\lambda\nu}$	$(1/g)k^{\lambda\nu}$
4	$(1/g)[(1+2K)h^{\lambda\nu}+{}^{(2)}k^{\lambda\nu}]$	$(1/g)[(1+2K)k^{\lambda\nu}+{}^{(3)}k^{\lambda\nu}]$
	$(4k \overset{\text{def}}{=} -{}^{(z)}k_\alpha{}^\alpha)$	

Proof. (15b) is equivalent to

$$*h^{\lambda\nu}=*g^{(\lambda\nu)}=\frac{1}{\mathfrak{G}}\frac{\partial\mathfrak{G}}{\partial h_{\lambda\nu}},\qquad *k^{\lambda\nu}=*g^{[\lambda\nu]}=\frac{1}{\mathfrak{G}}\frac{\partial\mathfrak{G}}{\partial k_{\lambda\nu}} \tag{21}$$

Substituting (11) and (17a) and (17b) into (21) successively and rearranging the range of the summations in view of (18) and Agreement 4.1, we have (19a) and (20a). The useful expressions (19b) and (20b) may be obtained by substituting suitable representation of $\underset{(s)}{X_\lambda{}^\nu}$, given in (14b), into (19a) and (20a), respectively. ■

As useful results of Theorem 4.4, we show in Table II the representations of the tensor $*g^{\lambda\nu}$ for the lower-dimensional cases $n=2,3,4$.

Remark 4.5. The expressions given in the Table II are, respectively, coincident with Chung's results for the case $n=2,3$ (Chung, 1979). The expression for $*h^{\lambda\nu}$ given in the table is also coincident with Hlavatý's result [(3.9a), Hlavatý, 1957, p. 8] for the case $n=4$.

Remark 4.6. It should be noted that the four-dimensional representation for $*k^{\lambda\nu}$ obtained in the present paper is more refined and useful than Hlavatý's result [(3.9b), Hlavatý, 1957, p. 8]:

$$*k^{\lambda\nu}=\frac{1}{\mathfrak{G}}\left(\mathfrak{H}k^{\lambda\nu}+\frac{\kappa}{2}\sqrt{\mathfrak{T}}\,E^{\omega\mu\lambda\nu}k_{\omega\mu}\right)\qquad\left(\kappa\overset{\text{def}}{=}\operatorname{sgn}E^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu}\right) \tag{22}$$

The coincidence of (22) with our result in Table II for $n=4$ follows from

$$\frac{\kappa}{2}\sqrt{\mathfrak{T}}\,E^{\omega\mu\lambda\nu}k_{\omega\mu}=\mathfrak{H}\left(2Kk^{\lambda\nu}+{}^{(3)}k^{\lambda\nu}\right)$$

which may be verified by using (2.10) (Hlavatý, 1957, pp. 6), (5), (7), and (4c).

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